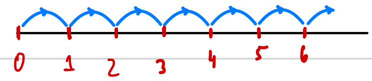
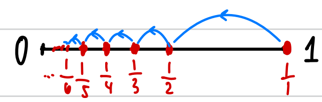
Math 451: Introduction to General Topology
Lecture 2

Examples of equinumerosity.



(a) Hilbert hotel: $\mathbb{N} \equiv \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. Indeed, $\mathbb{N} \to \mathbb{N}^+$ by $n \mapsto n+1$ is a bijection.



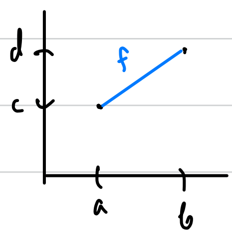
(b) $[0,1] \equiv [0,1)$. Similarly, $[0,1) \equiv (0,1)$ hence $[0,1] \equiv (0,1)$.

Proof 1. Carve a Hilbert hotel inside $[0,1]$ by taking any sequence $(x_n) \subseteq [0,1]$ of pairwise distinct elements, e.g. $x_n = \frac{1}{n}$, so $x_1 = 1$. Then map each x_n to x_{n+1} and map every $x \notin \{x_n : n \in \mathbb{N}^+\}$ to itself.
This is a bijection $[0,1] \to [0,1)$. □

Proof 2. Clearly $[0,1) \hookrightarrow [0,1]$ and also $[0,1] \hookrightarrow [0,1)$ by the scaling map with $\frac{1}{2}$.
Thus by Cantor–Schröder–Bernstein, $[0,1] \equiv [0,1)$. □

(c) $(a,b) \equiv (c,d)$, for all $a<b$ and $c<d$ in $\mathbb{R}$.

Proof.

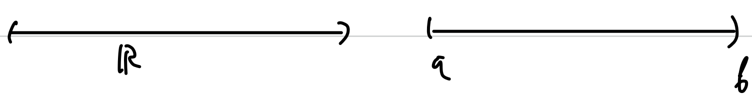


$f : (a,b) \to (c,d)$ linear, by $x \mapsto \frac{d-c}{b-a}(x-a) + c$.
This is clearly a bijection, its inverse is defined similarly.

(d) $\mathbb{R} \equiv (a,b)$.

Proof. It is enough to prove $\mathbb{R} \equiv (-1,1)$. Let $f : (-1,1) \to \mathbb{R}$

$$x \mapsto \frac{x}{1-|x|}.$$



One easily verifies that this is a bijection, for example by constructing a inverse. □

(e) $\mathcal{P}(X) \equiv 2^X$ for any set X. In general, for sets A and B, we denote by B^A the set of all functions from A to B. For example, $\mathbb{R}^3$ is the set of triples of reals, and each triple (x_0, x_1, x_2) is a function from $3 := \{0,1,2\}$ to $\mathbb{R}$

given by $i \mapsto x_i$. Similarly, a sequence (x_n) of reals is just a function $\mathbb{N} \to \mathbb{R}$ given by $n \mapsto x_n$. We denote the set of sequences by $\mathbb{R}^{\mathbb{N}}$.
Thus 2^X is the set of all 0-1 valued functions on X, in other words indicator functions of sets.

Proof. Define $f : \mathcal{P}(X) \to 2^X$ by mapping each $A \subseteq X$ to its indicator function: $\mathbb{1}_A : X \to \{0,1\} =: 2$ given by

$$\mathbb{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{o.w.} \end{cases}$$

Thus, we map $A \mapsto \mathbb{1}_A$.

This map f has an inverse $g : 2^X \to \mathcal{P}(X)$ given by $h \mapsto \{x \in X : h(x) = 1\}$. Thus f is a bijection. $\square$

In particular, $\mathcal{P}(\mathbb{N}) \cong 2^{\mathbb{N}}$, where $2^{\mathbb{N}}$ is the set of all binary (i.e. 0-1 valued) sequences over $\mathbb{N}$.

We already proved the following:

Prop. For any sets A, B, we have: $A \hookrightarrow B \overset{(AC)}{\iff} B \twoheadrightarrow A$.

Finite/infinite sets.

The set of natural numbers $\mathbb{N}$ is given by the set theory and it contains 0 which is just $\emptyset$. Each natural number $n \in \mathbb{N}$ is equal to $\{0, 1, \dots, n-1\}$, so $n < m \iff n \in m$.

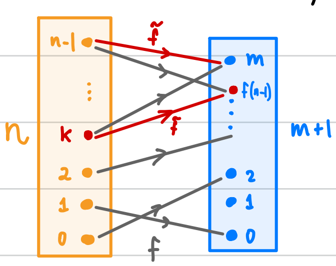
Def. A set X is called finite if it is equinumerous with a natural number, i.e. there is $n \in \mathbb{N}$ such that $X \cong n = \{0, 1, \dots, n-1\}$. It is called infinite otherwise.

Def. A set X is called Dedekind infinite if it is equinumerous with its proper subset, i.e. there is a proper subset $X' \subsetneq X$ such that $X \cong X'$ (e.g. $\mathbb{N} \cong \mathbb{N}^+$).

Otherwise, call X **Dedekind finite**.

Prop (Pigeonhole Principle). Finite sets are Dedekind finite. In fact, for all $n, m \in \mathbb{N}$, if $n \hookrightarrow m$ then $n \leq m$.

Proof. We prove by induction on m. For $m = 0 = \emptyset$, if $f : n \hookrightarrow \emptyset$ then $n = \emptyset = 0$.
Now suppose the statement is true for an arbitrary $m \in \mathbb{N}$ and prove for $m+1$.
Recall that $m = \{0, 1, \dots, m-1\}$ and $m+1 = \{0, 1, \dots, m\}$. Suppose $f : n \hookrightarrow m+1$.



If $m \notin f(\{0, 1, \dots, n-1\})$, then in fact $f : n \hookrightarrow m$, so $n \leq m < m+1$ by induction. Thus, assume $\exists\, k \in n$ s.t. $f(k) = m$.
Then define $\tilde{f} : n \to m+1$ by $\tilde{f}(i) = f(i)$ if $i \notin \{k, n-1\}$, and $\tilde{f}(k) := f(n-1)$ and $\tilde{f}(n-1) := f(k) = m$. Then $\tilde{f}$ is still injective, but it now maps $n-1$ to m, so the restriction of $\tilde{f}$ to $n-1 := \{0, 1, \dots, n-2\}$ is an injection of $n-1$ into $m = \{0, 1, \dots, m-1\}$. Hence by induction, $n-1 \leq m$, so $n \leq m+1$. $\square$

Theorem (AC)². For every set X, TFAE:

(1) X is infinite.

(2) X is Dedekind infinite.

(3) $\mathbb{N} \hookrightarrow X$.

(4) $X \twoheadrightarrow \mathbb{N}$.

Proof. (2) $\Rightarrow$ (1). This is the contrapositive of Pigeonhole Principle.

(3) $\overset{(AC)}{\Longleftrightarrow}$ (4). We already did, $\Leftarrow$ uses (AC).

(3) $\Rightarrow$ (2). We do the Hilbert hotel trick inside X. Indeed, because $\mathbb{N} \hookrightarrow X$, we assume WLOG that $\mathbb{N} \subseteq X$. Then define $f : X \to X \setminus \{0\}$ by

$$f(x) := \begin{cases} x & \text{if } x \notin \mathbb{N} \\ x+1 & \text{if } x \in \mathbb{N} \end{cases}.$$

This clearly a bijection, just like with the proof of $[0,1] \equiv [0,1)$.

(1) $\overset{(AC)}{\Rightarrow}$ (3). We define a function $f : \mathbb{N} \to X$ by defining $f(n)$ by induction/recursion

on $n \in \mathbb{N}$. For an arbitrary $n \in \mathbb{N}$, we suppose that $f(0), \dots, f(n-1)$ are already defined and we define $f(n) :=$ any element of $X \setminus \{f(0), \dots, f(n-1)\}$.
In other words: $f(0) :=$ choose some element of X, $f(1) :=$ choose some element of $X \setminus \{f(0)\}$, $f(2) :=$ choose an element of $X \setminus \{f(0), f(1)\}$, etc. To make this a rigorous definition, we need to apply AC and get a choice function $c : P(X) \setminus \{\emptyset\} \to X$ such that for each $A \subseteq X$, $c(A) \in A$. Then we can define

$$f(n) := c(X \setminus \{f(0), f(1), \dots, f(n-1)\}).$$

This is an injection by definition. □

HW: Prove that (2)⇒(3) directly without AC.

Countable sets. A set X is said to be countable (ctbl) if $X \equiv \mathbb{N}$ or X is finite.

Theorem (without AC). For a set X, TFAE:

(1) X is ctbl.

(2) $X \hookrightarrow \mathbb{N}$.

(3) $\mathbb{N} \twoheadrightarrow X$. (This means $\exists$ surjection $h : \mathbb{N} \to X$, so $X = h(\mathbb{N}) = \{h(n) : n \in \mathbb{N}\}$. We call this h an enumeration of X and commonly write $X = \{x_n : n \in \mathbb{N}\}$ where $x_n = h(n)$.)

Proof. (1) ⇒ (2). This is trivial since either $X \equiv \mathbb{N} \hookrightarrow \mathbb{N}$ or $X \equiv n = \{0, 1, \dots, n-1\} \subseteq \mathbb{N}$.

(2) ⇒ (3). Already done.

(3) ⇒ (2). We know this using AC but we don't need AC when the domain is $\mathbb{N}$ because we can use the well-ordering of $\mathbb{N}$ to define a right inverse of a surjection $f : \mathbb{N} \twoheadrightarrow X$. Indeed, map each $x \in X$ to the least natural number in $f^{-1}(\{x\})$.

(2) ⇒ (1). WLOG, assume $X \subseteq \mathbb{N}$ and infinite (if X is finite then we are done). Define a function $f : \mathbb{N} \to X$ by recursion, as follows: for an arbitrary $n \in \mathbb{N}$, suppose that $f(0), f(1), \dots, f(n-1)$ are defined and set $f(n) := \min X \setminus \{f(0), f(1), \dots, f(n-1)\}$, using the well-ordering of $\mathbb{N}$ and the fact that $X \setminus \{f(0), \dots, f(n-1)\}$ is nonempty since X is infinite. □